

# Extendable endomorphisms on factors

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## Abstract

We begin this note with a von Neumann algebraic version of the elementary but extremely useful fact about being able to extend inner-product preserving maps from a total set of the domain Hilbert space to an isometry defined on the entire domain. This leads us to the notion of when a ‘good’ endomorphism of a factorial probability space  $(M, \phi)$  admits a natural extension to an endomorphism of  $L^2(M, \phi)$ . We exhibit examples of such extendable endomorphisms.

We then pass to  $E_0$ -semigroups  $\alpha = \{\alpha_t : t \geq 0\}$  of factors, and observe that extendability of this semigroup (i.e., extendability of each  $\alpha_t$ ) is a cocycle-conjugacy invariant of the semigroup. We identify a necessary condition for extendability of such an  $E_0$ -semigroup, which we then use to show that the Clifford flow on the hyperfinite  $II_1$  factor is not extendable. As a consequence of this we point out an error in [ABS]

## 1 Some conventions

For any index set  $I$ , we write  $I^* = \bigcup_{n=0}^{\infty} I^n$  where  $I^0 = \emptyset$ , and

$$\mathbf{i} \vee \mathbf{j} = (i_1, \dots, i_m) \vee (j_1, \dots, j_n) = (i_1, \dots, i_m, j_1, \dots, j_n) \text{ whenever } \mathbf{i} = (i_1, \dots, i_m), \mathbf{j} = (j_1, \dots, j_n) \in I^*.$$

By a von Neumann probability space, we shall mean a pair  $(M, \phi)$  consisting of a von Neumann algebra and a normal state. For such an  $(M, \phi)$ , and an  $x \in M$ , we shall write  $\hat{x} = \lambda_M(x)\widehat{1}_M$  and  $\widehat{1}_M$  for the cyclic vector for  $\lambda_M(M)$  in  $L^2(M, \phi)$ .

By the (central) support of  $\phi$ , we shall mean the central projection  $z (= z_\phi)$  such that  $\ker(\lambda_M) = M(1 - z)$ . Clearly  $z_\phi = 1_M$  if  $M$  is a factor.

Finally, if  $\{x_i : i \in I\} \subset M$ , and  $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ , we shall write  $x_{\mathbf{i}} = x_{i_1} x_{i_2} \cdots x_{i_n}$ .

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## 2 An existence result

PROPOSITION 2.1. *Let  $(M_i, \phi_i), i = 1, 2$  be von Neumann probability spaces with  $z_i = z_{\phi_i}$ . Suppose  $S^{(j)} = \{x_i^{(j)} : i \in I\}$  is a set of self-adjoint elements which generates  $M_j$  as a von Neumann algebra, for  $j = 1, 2$ . (Note the crucial assumption that both the  $S^{(j)}$  are indexed by the same set.) Suppose*

$$\phi_1(x_i^{(1)}) = \phi_2(x_i^{(2)}) \quad \forall \mathbf{i} \in I^*. \quad (2.1)$$

*Then there exists a unique isomorphism  $\theta : M_1 z_1 \rightarrow M_2 z_2$  such that  $\phi_2 \circ \theta|_{M_1 z_1} = \phi_1|_{M_1 z_1}$  and  $\theta(x_i^{(1)} z_1) = x_i^{(2)} z_2 \quad \forall i \in I$ .*

*Proof.* The hypothesis implies that, for  $j = 1, 2$ , the set  $\{x_{\mathbf{i}}^{(j)} : \mathbf{i} \in I^*\}$  linearly spans a \*-subalgebra which is necessarily  $\sigma$ -weakly dense in  $M_j$ . Since  $\langle \widehat{x_{\mathbf{i}}^{(1)}}, \widehat{x_{\mathbf{j}}^{(1)}} \rangle = \langle \widehat{x_{\mathbf{i}}^{(2)}}, \widehat{x_{\mathbf{j}}^{(2)}} \rangle \quad \forall \mathbf{i}, \mathbf{j} \in I^*$ , there exists a unique unitary operator  $u : L^2(M_1, \phi_1) \rightarrow L^2(M_2, \phi_2)$  such that  $ux_{\mathbf{i}}^{(1)} = \widehat{x_{\mathbf{i}}^{(2)}} \quad \forall \mathbf{i} \in I^*$ .

Now observe that

$$\begin{aligned} u\lambda_{M_1}(x_{\mathbf{i}}^{(1)})u^* \widehat{x_{\mathbf{j}}^{(2)}} &= u\lambda_{M_1}(x_{\mathbf{i}}^{(1)})\widehat{x_{\mathbf{j}}^{(1)}} \\ &= ux_{\mathbf{i} \vee \mathbf{j}}^{(1)} \\ &= \widehat{x_{\mathbf{i} \vee \mathbf{j}}^{(2)}} \\ &= \lambda_{M_2}(x_{\mathbf{i}}^{(2)})\widehat{x_{\mathbf{j}}^{(2)}}; \end{aligned}$$

and hence that  $u\lambda_{M_1}(x_{\mathbf{i}}^{(1)})u^* = \lambda_{M_2}(x_{\mathbf{i}}^{(2)}) \quad \forall \mathbf{i} \in I$ .

On the other hand,  $\{x \in M_1 : u\lambda_{M_1}(x)u^* \in \lambda_{M_2}(M_2)\}$  is clearly a von Neumann subalgebra of  $M_1$ ; since this has been shown to contain each  $x_{\mathbf{i}}^{(1)}$ , we may deduce that this must be all of  $M_1$ . Now notice that  $L^2(M_j, \phi_j) = L^2(M_j z_j, \phi_j|_{M_j z_j})$ , that  $\lambda_{M_j}(x) = \lambda_{M_j z_j}(xz_j) \quad \forall x \in M_j$ , and that  $\lambda_{M_j z_j}$  maps  $M_j z_j$  isomorphically onto its image.

The proof is completed by defining

$$\theta(x) = \lambda_{M_2 z_2}^{-1}(u\lambda_{M_1}(x)u^*) \quad \forall x \in M_1 z_1.$$

□

REMARK 2.2. 1. In the proposition, even if it is the case that  $N := \{x_i^{(2)} : i \in I\}'' \subsetneq M_2$ , we can still apply the result to

$(N, \phi_2|_N)$  in place of  $(M_2, \phi_2)$  and deduce the existence of a normal homomorphism of  $M_1$  into  $M_2$  which sends  $x_i^{(1)}$  to  $x_i^{(2)}z$  for each  $i$  (and  $1_{M_1}$  to the projection  $z = z_{\phi_2|_N} \in N$ ).

2. In the special case that the  $N$  of the last paragraph is a factor, the  $z$  there is nothing but  $\text{id}_{M_2}$  and in particular, Proposition 2.1 can be strengthened as follows:

Let  $(M_j, \phi_j)$ ,  $j = 1, 2$  be von Neumann probability spaces. Suppose  $S^{(j)} = \{x_i^{(j)} : i \in I\} \subset M_j$  is a set of self-adjoint elements such that  $S^{(1)''} = M_1$  and  $S^{(2)''}$  is a factor  $N \subset M_2$ . Suppose

$$\phi_1(x_i^{(1)}) = \phi_2(x_i^{(2)}) \quad \forall i \in I^*. \quad (2.2)$$

Then there exists a unique normal  $*$ -homomorphism  $\theta : M_1 \rightarrow N \subset M_2$  such that  $\theta(x_i^{(1)}) = x_i^{(2)}$  for all  $i \in I$ .

**COROLLARY 2.3.** 1. If  $\theta_i$  is a  $\phi_i$ -preserving unital endomorphism of a von Neumann probability space  $(M_i, \phi_i)$ , for  $i \in \Lambda$ , then there exists:

- (a) an endomorphism  $\otimes_{i \in \Lambda} \theta_i$  of the tensor product  $(\otimes_{i \in \Lambda} M_i, \otimes_{i \in \Lambda} \phi_i)$  such that  $(\otimes_{i \in \Lambda} \theta_i)(\otimes_{i \in \Lambda} x_i) = z(\otimes_{i \in \Lambda} \theta_i(x_i)) \quad \forall x_i = x_i^* \in M_i$ ; and
- (b) an endomorphism  $*_{i \in \Lambda} \theta_i$  of the free product  $(*_{i \in \Lambda} M_i, *_{i \in \Lambda} \phi_i)$  such that  $(*_{i \in \Lambda} \theta_i)(\lambda(x_j)) = z\lambda(\theta_j(x_j)) \quad \forall x_j \in M_j$  where we simply write  $\lambda$  for each ‘left-creation representation’  $\lambda : M_j \rightarrow \mathcal{L}(*_{i \in \Lambda} L^2(M_i, \phi_i))$  for every  $j \in I$ .

In the above existence assertions, the symbol  $z$  represents an appropriate projection (= image of the identity of the domain of the endomorphism in question).

2. If each  $M_i$  above is a factor, then (the  $z$  in the above statement can be ignored, as it is the identity of the appropriate algebra) and all endomorphisms above are unital monomorphisms.

*Proof.* It is not hard to see that Remark 2.2(1) is applicable to  $S^{(1)} = \{\otimes_i x_i : x_i = x_i^* \in M_i, x_i = 1_{M_i} \text{ for all but finitely many } i\}$  and  $S^{(2)} = \{\otimes_i \theta_i(x_i) : x_i = x_i^* \in M_i, x_i = 1_{M_i} \text{ for all but finitely many } i\}$  (resp.,  $S^{(1)} = \{\lambda(x_i) : i \in \Lambda, x_i = x_i^* \in M_i, \phi_i(x_i) = 0\}$  and  $S^{(2)} = \{\lambda(\theta_i(x_i)) : i \in \Lambda, x_i = x_i^* \in M_i, \phi_i(x_i) = 0\}$ ).

The second fact follows from Remark 2.2(2) because normal endomorphisms of factors are unital isomorphisms onto their images, and the tensor (resp., free) product of factors is a factor.  $\square$

For later reference, the next lemma identifies what we called the ‘central support  $z_\phi$  of a normal state  $\phi$  on a von Neumann algebra’ in one simple special case (of a ‘vector-state’).

LEMMA 2.4. *Suppose  $N \subset \mathcal{L}(\mathcal{H})$  is a von Neumann algebra,  $\xi \in \mathcal{H}$  is a unit vector, and  $\phi$  is the vector state defined on  $N$  by  $\phi(x) = \langle x\xi, \xi \rangle$ . If  $\mathcal{H}_0 = \overline{N\xi}$ , then a candidate for ‘the GNS triple for  $(N, \phi)$ ’ is given by  $(\mathcal{H}_0, id_N|_{\mathcal{H}_0}, \xi)$ . In particular, the central support of  $\phi$  is given by the projection  $z = \wedge\{p \in N : \text{ran } p \supseteq \mathcal{H}_0\}$ .*

*Proof.* It is clear that  $\xi$  is a cyclic vector for  $N|_{\mathcal{H}_0}$  and the assertion regarding GNS triples follows. Hence if  $z \in \mathcal{P}(Z(N))$  is such that  $N(1 - z) = \ker id_N|_{\mathcal{H}_0}$ , then  $z = \wedge\{p \in \mathcal{P}(N) : p|_{\mathcal{H}_0} = (1_N)|_{\mathcal{H}_0}\}$ , i.e.,  $z = \wedge\{p \in \mathcal{P}(N) : \text{ran } p \supseteq \mathcal{H}_0\}$ .  $\square$

### 3 Extendable endomorphisms

For the remainder of this paper, we make the standing assumption that  $\phi$  is a faithful normal state on a factor  $M$ . We identify  $x \in M$  with  $\lambda_M(x)$ , and simply write  $J$  for the modular conjugation operator ( $J_\phi$ ). Recall, thanks to the Tomita-Takesaki theorem that  $j = J(\cdot)J$  is a \*-preserving conjugate-linear isomorphism of  $\mathcal{L}(L^2(M, \phi))$  onto itself, which maps  $M$  and  $M'$  onto one another, and that  $\widehat{1}_M$  is also a cyclic and separating vector for  $M'$ . We shall assume that  $\theta$  is a normal unital \*-endomorphism which preserves  $\phi$ . The invariance assumption  $\phi \circ \theta = \phi$  implies that there exists a unique isometry  $u_\theta$  on  $L^2(M, \phi)$  such that  $u_\theta x \widehat{1}_M = \theta(x) \widehat{1}_M$  and equivalently, that  $u_\theta x = \theta(x) u_\theta \forall x \in M$ .

DEFINITION 3.1. *If  $M, \phi, \theta$  are as above, and if the associated isometry  $u_\theta$  of  $L^2(M, \phi)$  commutes with the modular conjugation operator  $J (= J_\phi)$ , we shall simply say  $\theta$  is a **good** endomorphism of the factorial non-commutative probability space  $(M, \phi)$ .*

REMARK 3.2. We briefly contemplated the use of ‘equi-modular’ in place of what we have called ‘good’ to suggest ‘equivariance with respect to the modular operator (or homomorphism) but settled in favour of the less descriptive and more unimaginative (but easier to type) ‘good’. We are fairly confident that this is a meaningful and useful notion in general, although perhaps a hard one to verify in explicit type III examples, and therefore carry on this generality as long as possible, and do observe and later use the obvious fact that in case  $\phi$  is the tracial state on a  $II_1$  factor, the ‘goodness’ condition is automatically satisfied.

Furthermore, we could (and probably should) have discussed non Neumann ‘measure spaces’  $(M, \phi)$  comprising a von Neumann algebra equipped with a faithful normal semifinite weight, but did not, as the necessary amendments appeared to be fairly elementary, while the notation gets unnecessarily cumbersome and it was convenient to keep using  $\widehat{1_M}$ .

**THEOREM 3.3.** *Suppose  $\theta$  is a good endomorphism of a factorial non-commutative probability space  $(M, \phi)$ . Then,*

1. *The equation  $\theta' = j \circ \theta \circ j$  defines a unital normal \*-endomorphism of  $M'$  which preserves  $\phi' = \overline{\phi \circ j}$ ; and*
2. *We have an identification*

$$\begin{aligned} L^2(M', \theta') &= L^2(M, \phi) \\ \widehat{1_{M'}} &= \widehat{1_M} \\ u_{\theta'} &= u_\theta \end{aligned}$$

3. *there exists a unique endomorphism  $\theta^{(2)}$  of  $\mathcal{L}(L^2(M, \phi))$  satisfying*

$$\theta^{(2)}(xj(y)) = \theta(x)j(\theta(y))z, \quad \forall x, y \in M$$

where  $z = \wedge\{p \in (\theta(M) \cup \theta'(M'))'' : \text{ran}(p) \supset \{\widehat{\theta(x)} : x \in M\}\}$ .

*Proof.* 1. It is clear that  $\theta' = j \circ \theta \circ j$  is a unital normal linear \*-endomorphism of  $M'$  and that

$$\overline{\phi' \circ \theta'} = \overline{\phi'} \circ \theta' = (\phi \circ j) \circ (j \circ \theta \circ j) = (\phi \circ \theta) \circ j = \phi \circ j = \overline{\phi'},$$

thereby proving (1).

2. This follows from the facts that  $\widehat{1_M}$  is a cyclic and separating vector for  $M$  and hence also for  $M'$ , the definition of  $\phi'$  which guarantees that

$$\begin{aligned} \langle j(x)\widehat{1_{M'}}, j(y)\widehat{1_{M'}} \rangle &= \phi'(j(y)^*j(x)) \\ &= \phi'(j(y^*x)) \\ &= \overline{\phi(y^*x)} \\ &= \phi(x^*y) \\ &= \langle y\widehat{1_M}, x\widehat{1_M} \rangle \\ &= \langle Jx\widehat{1_M}, Jy\widehat{1_M} \rangle \\ &= \langle JxJ\widehat{1_M}, JyJ\widehat{1_M} \rangle \\ &= \langle j(x)\widehat{1_M}, j(y)\widehat{1_M} \rangle \end{aligned}$$

and the definitions of the ‘implementing isometries’, which show that

$$\begin{aligned}
u_{\theta'}(j(x)\widehat{1_{M'}}) &= \theta'(j(x))\widehat{1_{M'}} \\
&= j(\theta(x))\widehat{1_{M'}} \\
&= J\theta(x)J\widehat{1_M} \\
&= J\theta(x)\widehat{1_M} \\
&= Ju_\theta x\widehat{1_M} \\
&= u_\theta Jx\widehat{1_M} \\
&= u_\theta JxJ\widehat{1_M} \\
&= u_\theta j(x)\widehat{1_{M'}}.
\end{aligned}$$

3. Notice that if  $x, y \in M$ , then

$$\begin{aligned}
\langle \theta(x)J\theta(y)J\widehat{1_M}, \widehat{1_M} \rangle &= \langle \theta(x)J\theta(y)\widehat{1_M}, \widehat{1_M} \rangle \\
&= \langle \theta(x)Ju_\theta y\widehat{1_M}, \widehat{1_M} \rangle \\
&= \langle \theta(x)u_\theta Jy\widehat{1_M}, \widehat{1_M} \rangle \quad (\text{as } \theta \text{ is good}) \\
&= \langle u_\theta xJy\widehat{1_M}, \widehat{1_M} \rangle \\
&= \langle u_\theta xJy\widehat{1_M}, u_\theta \widehat{1_M} \rangle \\
&= \langle xJy\widehat{1_M}, \widehat{1_M} \rangle.
\end{aligned}$$

Set  $S^{(1)} = \{xj(y) : x = x^*, y = y^*, x, y \in M\}$ , and  $S^{(2)} = \{\theta(x)j(\theta(y)) : xj(y) \in S^{(1)}\}$ , and deduce from the factoriality of  $M$  that  $S^{(1)''} = \mathcal{L}(L^2(M, \phi))$ .

Now we wish to apply Remark 2.2(1) with  $N = S^{(2)''} = \theta(M) \vee j(\theta(M))$  (where, both here and in the sequel, we write  $A \vee B = (A \cup B)''$  for the von Neumann algebra generated by von Neumann algebras  $A$  and  $B$ ) and  $\phi_1 = \phi_2 = \langle (\cdot)\widehat{1_M}, \widehat{1_M} \rangle$ . For this, deduce from Lemma 2.4 that

$$\begin{aligned}
z &= \wedge\{p \in \mathcal{P}(N) : \text{ran } p \supset N\widehat{1_M}\} \\
&= \wedge\{p \in \mathcal{P}(N) : \text{ran } p \supset \{\theta(x)\widehat{1_M}, \theta'(j(x))\widehat{1_M} : x \in M\}\} \\
&= \wedge\{p \in \mathcal{P}(N) : \text{ran } p \supset \{\widehat{\theta(x)} : x \in M\}\}
\end{aligned}$$

and the proof of the Theorem is complete. □

**COROLLARY 3.4.** *Let  $\theta$  be a good endomorphism of a factorial non-commutative probability space  $(M, \phi)$  in standard form (i.e., viewed*

as embedded in  $\mathcal{L}(L^2(M, \phi))$  as above). The following conditions on  $\theta$  are equivalent:

1. there exists a unital normal  $*$ -endomorphism  $\theta^{(2)}$  of  $\mathcal{L}(L^2(M, \phi))$  such that  $\theta^{(2)}(x) = \theta(x)$  and  $\theta^{(2)}(j(x)) = j(\theta(x))$  for all  $x \in M$ .
2.  $N = \theta(M) \vee j(\theta(M))$  is a factor; and in this case,  $N$  is necessarily a type I factor.
3.  $\{\widehat{x\theta(y)} : x \in N', y \in \theta(M)\}$  is total in  $L^2(M)$ .

An endomorphism of a factor which satisfies the equivalent conditions above will be said to be **extendable**.

*Proof.* Deduce from the assumed factoriality of  $M$  that  $M_1 =: M \vee M' = \mathcal{L}(L^2(M, \phi))$ . Also set  $M_2 = \mathcal{L}(L^2(M, \phi))$ ,  $\phi_1 = \phi_2 = \langle (\cdot) \widehat{1}_M, \widehat{1}_M \rangle$ ,  $S^{(1)} = \{xj(y) : x, y \in M, x = x^*, y = y^*\}$ ,  $S^{(2)} = \{\theta(x)\theta'(j(y)) : x, y \in M, x = x^*, y = y^*\}$  and  $N = S^{(2)''} = \theta(M) \vee \theta'(M') = \theta(M) \vee j(\theta(M))$ .

(1)  $\Rightarrow$  (2) :

$$\begin{aligned} N &= \theta(M) \vee j(\theta(M)) \\ &= \theta^{(2)}(M) \vee \theta^{(2)}(j(M)) \\ &= \theta^{(2)}(M \vee j(M)) \\ &= \theta^{(2)}(\mathcal{L}(L^2(M, \phi))) ; \end{aligned}$$

the homomorphic image of a type I factor is also a type I factor, and an appeal to Theorem 3.3 and Corollary 2.3(2) completes the proof of (1)  $\Rightarrow$  (2).

(2)  $\Rightarrow$  (3) : Note that  $[N\hat{1}] = [\theta(M)j(\theta(M)\hat{1})] = [\theta(M)\hat{1}]$ , since  $\theta(a)J\theta(b)J\hat{1} = \theta(ab^*)\hat{1}$ , and hence,

$$\begin{aligned} [\{\widehat{x\theta(y)} : x \in N', y \in \theta(M)\}] &= [N'\theta(M)\hat{1}] \\ &= [N'N\hat{1}] \\ &= L^2(M) , \end{aligned}$$

since the only non-trivial subspace invariant under a factor and its commutant is the whole space. (We have used the symbol  $[S]$  above and in the sequel to denote the closed subspace spanned by a subset  $S$  of a Hilbert space. Similarly, if  $S \subset \mathcal{L}(\mathcal{H})$ , we shall write  $[S]$  to denote the closure, in the strong operator topology (equivalently, the

weak operator topology) of the linear subspace of  $\mathcal{L}(\mathcal{H})$  spanned by  $S.$ )

(3)  $\Rightarrow$  (1) : For any  $x \in N'$ , define  $T_x = xu_\theta$  (with  $u_\theta$  being the isometry on  $L^2(M)$  which ‘implements  $\theta'$ ’ and satisfies  $u_\theta x = \theta(x)u_\theta \forall x \in M$ ). Hence if  $x_1, x_2 \in N'$ , and  $y \in M$ , we have (since  $\theta(y) \in N$ )

$$\begin{aligned} T_{x_2}^* T_{x_1} y &= u_\theta^* x_2^* x_1 u_\theta y \\ &= u_\theta^* x_2^* x_1 \theta(y) u_\theta \\ &= u_\theta^* \theta(y) x_2^* x_1 u_\theta \\ &= y u_\theta^* x_2^* x_1 u_\theta \\ &= y T_{x_2}^* T_{x_1}. \end{aligned}$$

Similarly, since  $u_\theta$  also implements  $\theta'$  (and  $N = (\theta(M) \vee \theta'(M'))$ ), we see by an identical reasoning that also  $T_{x_2}^* T_{x_1}$  commutes with every  $y' \in M'$ . Hence it must be the case that  $T_{x_2}^* T_{x_1}$  is a scalar for every  $x_1, x_2 \in N'$ ; and that scalar is necessarily equal to  $\langle T_{x_2}^* T_{x_1} \hat{1}, \hat{1} \rangle = \langle x_1 \hat{1}, x_2 \hat{1} \rangle$ . Now deduce that if  $x_1, x_2 \in N', y_1, y_2 \in M$ , then

$$\begin{aligned} \langle x_1 \hat{1}, x_2 \hat{1} \rangle \langle y_1 \hat{1}, y_2 \hat{1} \rangle &= \langle T_{x_2}^* T_{x_1} y_1 \hat{1}, y_2 \hat{1} \rangle \\ &= \langle T_{x_1} y_1 \hat{1}, T_{x_2} y_2 \hat{1} \rangle \\ &= \langle x_1 u_\theta y_1 \hat{1}, x_2 u_\theta y_2 \hat{1} \rangle \\ &= \langle x_1 \theta(y_1) \hat{1}, x_2 \theta(y_2) \hat{1} \rangle; \end{aligned}$$

deduce the existence of a unique unitary operator  $w : [N' \hat{1}] \otimes L^2(M) \rightarrow L^2(M)$  satisfying

$$w(x \hat{1} \otimes y \hat{1}) = x \theta(y) \hat{1}$$

for all  $x \in N', y \in M$ . Now the equation

$$\theta^{(2)}(z) = w(1 \otimes z)w^*$$

defines the desired endomorphism of  $\mathcal{L}(L^2(M))$  (which extends both  $\theta$  and  $\theta'$ ). Indeed,  $m \in M, x \in N', y \in M$ , then we have

$$\begin{aligned} w(1 \otimes m)w^* x \theta(y) \hat{1} &= w(x \hat{1} \otimes my \hat{1}) \\ &= x \theta(my) \hat{1} \\ &= \theta(m)x \theta(y) \hat{1} \quad (\text{since } x \in N'), \end{aligned}$$

so  $\theta^{(2)}(m) = \theta(m)$ , for all  $m \in M$ . Now to prove that  $\theta^{(2)}(m') = \theta'(m')$  it is enough, by appealing to an identical reasoning, to verify that

$$w(x \hat{1} \otimes y' \hat{1}) = x \theta'(y') \hat{1},$$

for  $x \in N'$  and  $y' \in M'$ . As  $\hat{1}$  is a cyclic vector for  $M$ , we may find  $y_i \in M$  such that  $y'\hat{1} = \lim y_i\hat{1}$ . Observe that

$$\begin{aligned} w(x\hat{1} \otimes y'\hat{1}) &= w(x\hat{1} \otimes \lim y_i\hat{1}) \\ &= \lim w(x\hat{1} \otimes y_i\hat{1}) \\ &= \lim x\theta(y_i)\hat{1} \\ &= \lim xu_\theta y_i\hat{1} \\ &= xu_\theta y'\hat{1} \\ &= xu_\theta y'\hat{1} \text{ since } u_\theta = u_{\theta'} \\ &= x\theta'(y')\hat{1}, \end{aligned}$$

as desired.  $\square$

## 4 Examples of Extendable Endomorphism

Note that any automorphism on a factor is extendable, since the conditions in Corollary 3.4 are satisfied.

Let  $\mathcal{R}$  denote the hyperfinite  $II_1$  factor and  $M$  be any  $II_1$  factor which is also a McDuff factor; i.e.,  $M \otimes \mathcal{R} \cong M$ . Let  $\alpha : M \otimes \mathcal{R} \mapsto M$  be an isomorphism and  $\beta : M \mapsto M \otimes \mathcal{R}$  be the monomorphism defined by  $\beta(m) = m \otimes 1$ , for  $m \in M$ . Let us write  $\theta = \beta \circ \alpha$ . so  $\theta$  is an endomorphism of  $M \otimes \mathcal{R}$  such that  $\theta(M \otimes \mathcal{R}) = M \otimes 1$ . As  $M \otimes \mathcal{R}$  is a  $II_1$  factor, the endomorphism  $\theta$  is necessarily good (see remark 3.2). Now by corollary 3.4, showing that  $\theta$  is extendable is equivalent to showing that  $\{\theta(M \otimes \mathcal{R}) \vee J\theta(M \otimes \mathcal{R})J\}$  is a type  $I$  factor, where  $J$  is the modular conjugation of  $M \otimes \mathcal{R}$ , which, of course, is  $J_M \otimes J_R$ . Note that

$$\begin{aligned} \{\theta(M \otimes \mathcal{R}) \vee J\theta(M \otimes \mathcal{R})J\} &= \{M \otimes 1 \vee J(M \otimes 1)J\} \\ &= \{M \otimes 1 \vee J_M M J_M \otimes 1\} \\ &= \mathcal{B}(L^2(M)) \otimes 1 \end{aligned}$$

So  $\{\theta(M \otimes \mathcal{R}) \vee J\theta(M \otimes \mathcal{R})J\}$  is a type  $I$  factor. That is  $\theta$  is extendable.

## 5 Extendability for $E_0$ -semigroups

**DEFINITION 5.1.**  $\{\alpha_t : t \geq 0\}$  is said to be an  $E_0$ -semigroup on a von Neumann probability space  $(M, \phi)$  if:

1.  $\alpha_t$  is a  $\phi$ -preserving normal unital \*-homomorphism of  $M$  for each  $t \geq 0$ ;

2.  $\alpha_0 = id_M$ ; and
3.  $[0, \infty) \ni t \mapsto \rho(\alpha_t(x))$  is continuous for each  $x \in M, \rho \in M_*$ .

**PROPOSITION 5.2.** Suppose  $\alpha = \{\alpha_t : t \geq 0\}$  is an  $E_0$ -semigroup on a factorial non-commutative probability space  $(M, \phi)$  and suppose  $\alpha_t$  is (good and) extendable for each  $t$ . We shall say, in such a case, that the  $E_0$ -semigroup  $\alpha$  is extendable. Then:

1. The equation  $\alpha'_t(x') = j(\alpha_t(j(x'))$  defines an  $E_0$ -semigroup on  $(M', \phi')$ , where  $\phi'(x') = \omega_{\widehat{1_M}}(x') = \langle x' \widehat{1_M}, \widehat{1_M} \rangle$ ; and
2. There exists a unique  $E_0$ -semigroup  $\{\alpha_t^{(2)} : t \geq 0\}$  on  $(\mathcal{L}(L^2(M, \phi)), \omega_{\widehat{1_M}})$  such that  $\alpha_t^{(2)}(xx') = \alpha_t(x)\alpha'_t(x') \forall x \in M, x' \in M'$ .

*Proof.* Existence of the endomorphisms  $\alpha'_t$  and  $\alpha_t^{(2)}$  is guaranteed by Corollary 3.4, so only the assertions regarding the semigroup property and continuity of the semigroups  $\{\alpha'_t : t \geq 0\}$  and  $\{\alpha_t^{(2)} : t \geq 0\}$  need justification.

The equation  $\alpha'_t = j \circ \alpha_t \circ j$  shows that  $\{\alpha'_t : t \geq 0\}$  inherits the property of being an  $E_0$ -semigroup from that of  $\{\alpha_t : t \geq 0\}$ .

The corresponding property for  $\{\alpha_t^{(2)} : t \geq 0\}$  is now seen to follow easily from the uniqueness assertion in Corollary 3.4(1).  $\square$

Before proceeding further, it might be wise to insert a remark which will repeatedly be found to be of use.

**REMARK 5.3.** Let  $(M, \phi)$  be a von Neumann probability space (i.e.,  $M$  is a von Neumann algebra and  $\phi$  is a (usually faithful) normal state on  $M$ ). Given a normal endomorphism  $\theta$  of  $M$ , we shall write  ${}_\theta L^2(M, \phi)$  for the Hilbert space  $L^2(M, \phi)$  with  $M$ -action given by  $x \cdot {}_\theta \xi = \theta(x)\xi$ . Given normal endomorphisms  $\theta_i, i = 1, 2$ , we shall need to consider the space  $(\theta_1, \theta_2)$  of intertwiners given by

$$(\theta_1, \theta_2) = \{T \in \mathcal{L}(L^2(M, \phi)) : T\theta_1(x) = \theta_2(x)T \forall x \in M\}.$$

We shall specifically be interested in  $E^\theta = (id_M, \theta)$  for any normal endomorphism  $\theta$  of  $M$ . It is obvious from the definitions that  $[(E^\theta)^* E^\theta] \subseteq M'$  and  $[(E^\theta)^* (E^\theta)^*] \subseteq \theta(M)'$  where primes denote commutants in  $\mathcal{L}(L^2(M, \phi))$ .

Let  $\alpha = \{\alpha_t : t \geq 0\}$  be an  $E_0$ -semigroup on a factorial non-commutative probability space  $(M, \phi)$ , and suppose  $\alpha_t$  is good for

each  $t$ . Assume  $M$  is acting standardly on  $\mathcal{H} = L^2(M, \phi)$ . As in Remark 5.3, we consider, for every  $t > 0$ , the interwiner space

$$E^{\alpha_t} = \{T \in \mathcal{B}(\mathcal{H}) : Tx = \alpha_t(x)T, \forall x \in M\}. \quad (5.3)$$

Let  $J$  be the modular conjugation operator on  $L^2(M)$ . Then  $j = J(\cdot)J$  defines an antilinear  $*$ -isomorphism of  $M$  onto  $M'$  and

$$\alpha' = \{\alpha'_t = j \circ \alpha_t \circ j : t \geq 0\}$$

defines an  $E_0$ -semigroup on the commutant  $M'$ ; and we have

$$E^{\alpha'_t} = \{T \in \mathcal{B}(\mathcal{H}) : Tx = \alpha'_t(x')T', \forall x' \in M'\}. \quad (5.4)$$

We first focus on the ‘fundamental unit’  $\{u_t : t \geq 0\}$  - which will establish the fact that  $E^{\alpha_t} \cap E^{\alpha'_t} \neq \emptyset \forall t \geq 0$ . For  $t \geq 0$ , the fact that ‘ $\phi$ ’ is preserved by  $\alpha_t$  implies the existence of a unique family (necessarily a one-parameter semigroup)  $\{u_t : t \geq 0\}$  of isometries on  $L^2(M)$  such that  $u_t x \hat{1} = \alpha_t(x) \hat{1} \forall x \in M$ , and consequently  $u_t \in E^{\alpha_t}$ . As  $\alpha_t$  is a good  $*$ -endomorphism of  $M$ , it follows - see Theorem 3.3(2) - that  $u_t$  also ‘implements  $\alpha'_t$ , i.e., also  $u_t x' \hat{1} = \alpha'_t(x') \hat{1} \forall x' \in M'$ , and consequently that  $u_t \in E^{\alpha'_t}$ . Thus,

$$u_t \in E^{\alpha_t} \cap E^{\alpha'_t} \forall t \geq 0. \quad (5.5)$$

For the sake of completeness, we state below a result that is already contained in [Alev], because it is re-stated in language that we find convenient (and are comfortable with).

**PROPOSITION 5.4.** *For  $t > 0$  ,  $(E^{\alpha_t}, (id, \mathcal{H}), (id, \mathcal{H}))$  is a Hilbert von Neumann  $M'$  -  $M'$  bimodule (in the sense of [BMSS]).*

*Proof.* It follows from equation 5.3 that  $E^{\alpha_t}$  is a weakly closed subspace of  $\mathcal{B}(\mathcal{H})$ . So, we only have to verify that  $[E^{\alpha_t*} E^{\alpha_t}] = M'$  and  $M' \subset [E^{\alpha_t} E^{\alpha_t*}]$ . Note that  $[X^*Y]$  means the strong, equivalently the closure in the weak (equivalently, strong) operator topology of the linear subspace spanned by  $\{x^*y : x \in X, y \in Y\}$ .

The inclusion

$$[E^{\alpha_t*} E^{\alpha_t}] \subset M' \quad (5.6)$$

is seen from Remark 5.3.

It is clear that if  $y' \in M'$  and  $T \in E_t^\alpha$ , then

$$\begin{aligned} \alpha_t(m)Ty' &= Tmy' \\ &= Ty'm \end{aligned}$$

for all  $m \in M$ . So,  $Ty' \in E_t^\alpha$  for all  $y' \in M'$  and  $T \in E_t^\alpha$  and hence,

$$E^{\alpha t} M' \subset E^{\alpha t}. \quad (5.7)$$

We may conclude that  $[E^{\alpha t *} E^{\alpha t}]$  is a weakly closed two sided ideal in  $M'$ . Being a factor,  $M'$  does not contain any non-trivial weakly closed proper ideal and consequently we may deduce that  $[E^{\alpha t *} E^{\alpha t}] = M'$  (since  $0 \neq 1 = u_t^* u_t \in E^{\alpha t *} E^{\alpha t}$  by equation 5.5).

To conclude the proof, we only need to verify that  $M' \subset [E^{\alpha t} E^{\alpha t *}]$ . For this, first notice, from Remark 5.3, that

$$[E^{\alpha t} E^{\alpha t *}] \subset \alpha_t(M)', \quad (5.8)$$

From equation 5.7, we observe that  $[E^{\alpha t} E^{\alpha t *}]$  is a weakly closed two sided ideal of  $\alpha_t(M)'$ . But we know that  $\alpha_t(M)'$  is a factor, so, it does not contain any proper weakly closed ideal and consequently we may deduce (from  $0 \neq e_t = u_t u_t^* \in E^{\alpha t} E^{\alpha t *}$ ) that

$$[E^{\alpha t} E^{\alpha t *}] = \alpha_t(M)'$$

Since  $M' \subset \alpha_t(M)'$ , we may conclude that  $M' \subset [E^{\alpha t} E^{\alpha t *}]$ .  $\square$

The next Proposition follows immediately from Proposition 5.4 (with  $\alpha'$  in place of  $\alpha$ ).

**PROPOSITION 5.5.** *For  $t > 0$ ,  $(E^{\alpha'_t}, (id, \mathcal{H}), (id, \mathcal{H}))$  is a Hilbert von Nuemann  $M - M$  bimodule.*

We will need the following explicit description of this bimodule.

**THEOREM 5.6.**  *$E^{\alpha_t} = [M'u_t] = \alpha_t(M)'u_t$ , where, of course,  $[M'u_t]$  denotes the weak operator closure of  $M'u_t$ .*

*Proof.* We already know that  $E^{\alpha_t}$  is Hilbert von Neumann  $M' - M'$ -bimodule. In particular  $E^{\alpha_t}$  is Hilbert von Neumann  $M'$  module. Now we shall verify that  $[M'u_t]$  is Hilbert von Neumann submodule of  $E^{\alpha_t}$ . For that we need to check that  $[M'u_t]$  is Hilbert von Neumann  $M'$  module and  $[M'u_t] \subset E^{\alpha_t}$ . For the first assertion notice that

$$\begin{aligned} \{(m'_1 u_t)^* m'_2 u_t : m'_1, m'_2 \in M'\} &= [u_t^* m'_1 m'_2 u_t] \\ &= [u_t^* M' u_t], \end{aligned}$$

so it suffices to check that  $u_t^* M' u_t \subset M'$ , i.e., that  $u_t^* m' u_t x = x u_t^* m' u_t \forall m' \in M', x \in M$ ; but

$$\begin{aligned} u_t^* m' u_t x &= u_t^* m' \alpha_t(x) u_t \quad (\text{since } u_t \in E_t^\alpha) \\ &= u_t^* \alpha_t(x) m' u_t \\ &= (\alpha_t(x^*) u_t)^* m' u_t \\ &= (u_t x^*)^* m' u_t \\ &= x u_t^* m' u_t. \end{aligned}$$

Conversely,  $m' = u_t^* u_t m' = u_t^* \alpha'_t(m') u_t$  so  $M' \subset u_t^* \alpha'_t(M') u_t \subset u_t^* M' u_t$ , and hence we do have  $M' = u_t^* M' u_t$ .

For the second assertion observe that

$$\begin{aligned} \alpha_t(m) m' u_t &= m' \alpha_t(m) u_t \\ &= m' u_t m, \end{aligned}$$

for all  $m \in M$  and  $m' \in M'$ , thus showing that  $M' u_t \subset E^{\alpha_t}$ , and hence also that  $[M' u_t] \subset E^{\alpha_t}$ .

Now suppose that there exist  $T \in E^{\alpha_t}$  such that  $T \in [M' u_t]^\perp$ , i.e.,  $T^* m' u_t = 0$  for all  $m' \in M'$ . Now notice that  $T^* m' u_t \hat{1}_M = T^* m' \hat{1}_M = T^* \hat{m}'$ , and hence conclude that  $T = 0$ . Deduce then from the Riesz lemma that  $E^{\alpha_t} = [M' u_t]$ .

Observe next that for  $m \in M$  and  $x \in \alpha_t(M)'$ , we have

$$\begin{aligned} \alpha_t(m) x u_t &= x \alpha_t(m) u_t \\ &= x u_t m, \end{aligned}$$

and deduce that  $\alpha_t(M)' u_t \subset E^{\alpha_t}$ . On the other if  $T \in E^{\alpha_t}$  observe that

$$\begin{aligned} T &= T u_t^* u_t \\ &= y u_t \end{aligned}$$

where  $y = T u_t^* \subset [E^{\alpha_t} E^{\alpha_t*}] = \alpha_t(M)'$ . That is  $T \in \alpha_t(M)' u_t$ . So we have  $E^{\alpha_t} \subset \alpha_t(M)' u_t$ , yielding  $E^{\alpha_t} = \alpha_t(M)' u_t$ , as desired.  $\square$

**REMARK 5.7.** 1. We have already seen that  $E^{\alpha_t}$  is Hilbert von Neumann  $M' - M'$ -bimodule, so  $[M' u_t]$  and  $\alpha_t(M)' u_t$  are also a Hilbert von Neumann  $M' - M'$ -bimodule.

2. Replacing  $\alpha'$  by  $\alpha$  and  $M'$  by  $M$  in Proposition 5.6, we get  $E_t^{\alpha'} = [Mu_t] = \alpha'_t(M)' u_t$ .

Writing  $M_1(t) = \alpha'_t(M')$  and  $M'_1(t) = \alpha_t(M)$ , we may summarize thus:

$$E^{\alpha t} = [M' u_t] = M'_1(t) u_t, \quad (5.9)$$

and

$$E_t^{\alpha'} = [Mu_t] = M_1(t) u_t. \quad (5.10)$$

Now for every  $t > 0$ , let us write  $H(t) = E^{\alpha t} \cap E^{\alpha t}$ . In fact,  $H(t)$  is actually a Hilbert space; if  $S, T \in H_t$  and  $x \in M, x' \in M'$ , we have:

$$T^* S x = T^* \alpha_t(x) S = (\alpha_t(x^*) T)^* S = (Tx^*)^* S = x T^* S,$$

since  $S, T \in E^{\alpha t}$  and

$$T^* S x' = T^* \alpha'_t(x') S = (\alpha'_t(x'^*) T)^* S = (Tx'^*)^* S = x' T^* S,$$

since  $S, T \in E_t^{\alpha'}$ . So  $T^* S$  commutes with both  $M$  and  $M'$  and as  $M$  is factor, we find that  $T^* S$  is a scalar multiple of the identity and the value of that scalar defines an inner product by way of

$$T^* S = \langle S, T \rangle I.$$

**PROPOSITION 5.8.** *Let  $\alpha = \{\alpha_t : t \geq 0\}$  be an  $E_0$ -semigroup on a factorial non-commutative probability space  $(M, \phi)$ , and suppose  $\alpha_t$  is good for each  $t$ . Suppose  $M$  is acting standardly on  $\mathcal{H} = L^2(M)$ . The following conditions on  $\alpha$  are equivalent.*

1.  $\alpha$  is extendable.
2. There exists a family  $\{u_n(t) : n \in I\} \subset M_1(t) \cap M'_1(t)$ , with  $I$  being either a singleton (in case  $\alpha_t^{(2)}$  is an automorphism for some, equivalently for all,  $t > 0$ ) or  $I$  is countably infinite (in case  $\alpha_t^{(2)}$  is not surjective for some, equivalently for all,  $t$ ) such that  $\{u_n(t)u_n^*(t) : n \in I\}$  is a family of orthogonal projections in  $M_1(t) \cap M'_1(t)$ , with  $\sum_{n \in I} u_n(t)u_n^*(t) = 1$  and  $u_n^*(t)u_n(t) = e_t := u_t u_t^* \forall t$ .

*Proof.* (1)  $\Rightarrow$  (2) : Suppose  $\alpha$  is exendable, with associated  $E_0$ -semigroup  $\alpha^{(2)} = \{\alpha_t^{(2)} : t > 0\}$  on  $\mathcal{B}(L^2(M))$ . For every  $t > 0$ ,  $\alpha_t^{(2)} : \mathcal{B}(L^2(M)) \mapsto \mathcal{B}(L^2(M))$  is an endomoprphism. So one can write  $\alpha_t^{(2)}(x) = \sum_{n \in I} v_n(t)xv_n(t)^*$  (with  $I$  as in the statement of the Proposition), where  $x \in \mathcal{B}(L^2(M))$  and  $\{v_n(t)\}_{n \in I}$  are isometries having mutually orthogonal ranges and  $\sum v_n(t)v_n(t)^* = 1$ . We

observe that  $v_n(t) \in H(t)$ . Now consider  $u_n(t) = v_n(t)u_t^*$ . Then observe that  $u_n(t) \in M_1(t) \cap M_1(t)'$ ,

$$\begin{aligned} u_n(t)^*u_n(t) &= u_t v_n(t)^*v_n(t)u_t^* \\ &= u_t u_t^* \quad (\text{since } v_n(t)^*v_n(t) = 1), \end{aligned}$$

and

$$\begin{aligned} \sum u_n(t)u_n(t)^* &= \sum v_n(t)u_t^*u_t v_n(t)^* \\ &= \sum v_n(t)v_n(t)^* \\ &= 1. \end{aligned}$$

(2)  $\Rightarrow$  (1) : Consider  $v_n(t) = u_n(t)u_t$ , and observe that

$$\begin{aligned} v_n(t)^*v_n(t) &= u_t^*u_n(t)^*u_n(t)u_t \\ &= u_t^*u_t u_t^*u_t \\ &= 1. \end{aligned}$$

and

$$\begin{aligned} \sum v_n(t)v_n(t)^* &= \sum u_n(t)u_t u_t^*u_n(t)^* \\ &= \sum u_n(t)u_n(t)^*u_n(t)u_n(t)^* \\ &= \sum u_n(t)u_n(t)^* \\ &= 1 \end{aligned}$$

Observe that, by definition,  $v_n(t) \in M_1(t)u_t \cap M_1(t)'u_t = E_t^{\alpha t} \cap E_t^{\alpha'}$  ( From 5.9 and 5.10 ). Consider the mapping  $\alpha_t^{(2)} : \mathcal{B}(L^2(M)) \mapsto \mathcal{B}(L^2(M))$  defined by  $\alpha_t^{(2)}(x) = \sum v_n(t)xv_n(t)^*$ , where  $x \in \mathcal{B}(L^2(M))$ . Now deduce from the properties of  $\{v_n(t) : n \in I\}$  that  $\alpha_t^{(2)}$  is indeed an endomorphism of  $\mathcal{B}(L^2(M))$ , and observe that

$$\begin{aligned} \alpha_t^{(2)}(m) &= \sum v_n(t)m v_n(t)^* \\ &= \sum \alpha_t(m)v_n(t)v_n(t)^* \quad (\text{since } v_n(t) \in E_t^{\alpha t}) \\ &= \alpha_t(m) \sum v_n(t)v_n(t)^* \\ &= \alpha_t(m), \end{aligned}$$

where  $m \in M$ . So we have  $\alpha_t^{(2)}(m) = \alpha_t(m)$ , for all  $m \in M$ . Since  $v_n(t) \in E_t^{\alpha'}$ , it can be shown similarly that  $\alpha_t^{(2)}(m') = \alpha_t'(m')$ , for all  $m' \in M'$ . So  $\alpha$  is indeed extendable.  $\square$

LEMMA 5.9. Let  $\alpha = \{\alpha_t : t \geq 0\}$  be an  $E_0$ -semigroup on a factorial non-commutative probability space  $(M, \phi)$  and suppose  $\alpha_t$  is good for each  $t$ . Suppose  $M$  is acting standardly on  $\mathcal{H} = L^2(M)$ . If  $\alpha$  is extendable then  $H = \{(t, T) : t > 0, T \in H(t)\}$  is a product system (in the sense of [Arv]) with the family of unitary maps  $\{u_{st} : s, t > 0\}$ , where

$$u_{st} : H(t) \otimes H(s) \mapsto H(s+t),$$

being given by  $u_{st}(T \otimes S) = TS$ , for  $T \in H(t), S \in H(s)$ .

*Proof.* Let  $\alpha^{(2)} = \{\alpha_t^{(2)} : t > 0\}$  be the extension of  $\alpha$  on  $\mathcal{B}(L^2(M))$ . Thus,  $\alpha^{(2)}$  is an  $E_0$ -semigroup on  $\mathcal{B}(L^2(M))$  satisfying  $\alpha_t^{(2)}(m) = \alpha_t(m)$  and  $\alpha_t^{(2)}(m') = \alpha'_t(m')$  for all  $m \in M$  and for all  $m' \in M'$ .

For every  $t > 0$ , let us consider the set

$$\mathcal{E}(t) = \{T \in \mathcal{B}(L^2(M)) : \alpha_t^2(x)T = Tx, \text{ for all } x \in \mathcal{B}(L^2(M))\}.$$

We shall write  $\mathcal{E} = \{(t, T) : T \in \mathcal{E}(t)\}$ ; then  $\mathcal{E}$  is a product system (see [Arv]). First, observe that  $H(t) = \mathcal{E}(t)$  for every  $t > 0$ . Indeed, if  $T \in H(t) = E^{\alpha_t} \cap E_t^{\alpha'}$ , then  $\alpha_t(m)T = Tm$  for all  $m \in M$  and  $\alpha'_t(m')T = Tm'$  for all  $m' \in M'$ . So it is clear that  $\alpha_t^{(2)}(x)T = Tx$  for all  $x \in M \cup M'$  and hence also for all  $x \in (M \vee M') = \mathcal{B}(L^2(M))$ . So,  $T \in \mathcal{E}(t)$ , and  $H(t) \subset \mathcal{E}(t)$ . The reverse inclusion is immediate from the definition  $\alpha_t^{(2)}$ . So we have  $H(t) = \mathcal{E}(t)$  and clearly  $H$  is a product system, since  $\mathcal{E}$  is!  $\square$

Fix some (any) unit vector  $\xi$  in  $L^2(M)$ . Observe, by definition of the inner product in  $H(t)$ , that the map  $H(t) \ni T \mapsto T\xi \in H(t)\xi$  is a unitary operator.

COROLLARY 5.10. If  $\alpha$  is extendable then  $H\xi = \{(t, \eta) : t > 0, \eta \in H(t)\xi\}$  is a product system with the family of maps  $\{v_{st} : s, t > 0\}$ , where

$$v_{st} : H(t)\xi \otimes H(s)\xi \mapsto H(t+s)\xi,$$

is given by  $v_{st}(T\xi \otimes S\xi) = TS\xi$ , for  $S \in H_s, T \in H_t$ .

*Proof.* We observe that for every  $s, t > 0$ ,  $v_{st}$  is unitary map. We have,

$$\begin{aligned} & \langle v_{st}(T_1\xi \otimes S_1\xi), v_{st}(T_2\xi \otimes S_2\xi) \rangle \\ &= \langle T_1S_1\xi, T_2S_2\xi \rangle \\ &= \langle S_2^*T_2^*T_1S_1\xi, \xi \rangle \\ &= \langle T_1, T_2 \rangle \langle S_1, S_2 \rangle \langle \xi, \xi \rangle, (\text{since } T_2^*T_1 = \langle T_1, T_2 \rangle 1 \text{ and } S_2^*S_1 = \langle S_1, S_2 \rangle 1) \\ &= \langle T_1\xi, T_2\xi \rangle \langle S_1\xi, S_2\xi \rangle, (\text{since } \langle \xi, \xi \rangle = 1) \\ &= \langle T_1\xi \otimes S_1\xi, T_2\xi \otimes S_2\xi \rangle \end{aligned}$$

where  $T_1, T_2 \in H(t)$  and  $S_1, S_2 \in H(s)$ . To prove that  $v_{st}$  is an unitary operator it remains to show that  $\{TS\xi : T \in H(t), S \in H(s)\}$  is a total set in  $H(t+s)\xi$ . Since  $\alpha$  is extendable, so from the Lemma 5.9 we have,  $\{TS : T \in H(t), S \in H(s)\}$  is total in  $H(t+s)\xi$ . Therefore  $\{TS\xi : T \in H(t), S \in H(s)\}$  is total in  $H(t+s)$ . So  $H\xi$  is a product system, since other details follow from the Lemma 5.9 immediately.

□

Now recall that  $E_0$ -semigroups  $\{\alpha_t : t \geq 0\}$  and  $\{\beta_t : t \geq 0\}$  of a von Neumann probability space  $(M, \phi)$  are said to be **cocycle conjugate** if there exists a weakly continuous family  $\{u_t : t \geq 0\}$  of unitary elements of  $M$  such that

1.  $u_{t+s} = u_s \alpha_s(u_t)$ ; and
2.  $\beta_t(x) = u_t \alpha_t(x) u_t^*$  for all  $x \in M$  and  $s, t \geq 0$ .

In such a case, we shall simply write

$$\{u_t : t \geq 0\} : \{\alpha_t : t \geq 0\} \simeq \{\beta_t : t \geq 0\}.$$

**PROPOSITION 5.11.** *Suppose  $\alpha = \{\alpha_t : t \geq 0\}$  and  $\beta = \{\beta_t : t \geq 0\}$  are cocycle conjugate  $E_0$  semigroups on a factorial probability space  $(M, \phi)$ , with*

$$\{u_t : t \geq 0\} : \{\alpha_t : t \geq 0\} \simeq \{\beta_t : t \geq 0\}.$$

*Then*

1.  $\{j(u_t) : t \geq 0\} : \{\alpha'_t : t \geq 0\} \simeq \{\beta'_t : t \geq 0\}$ .
2. *If each  $\alpha_t$  is extendable, so is each  $\beta_t$ ; and in fact,*

$$\{u_t j(u_t) : t \geq 0\} : \{\alpha_t^{(2)} : t \geq 0\} \simeq \{\beta_t^{(2)} : t \geq 0\}.$$

*Proof.* 1) Observe, to start with, that if  $\theta$  is an endomorphism of  $(M, \phi)$ , then, by definition, we have  $\theta' = j \circ \theta \circ j$ , i.e.,  $j \circ \theta = \theta' \circ j$ . Hence, writing  $u'_t = j(u_t)$ , we have

$$\begin{aligned} u'_{s+t} &= j(u_s \alpha_s(u_t)) \\ &= j(u_s) j(\alpha_s(u_t)) \\ &= j(u_s) \alpha'_s(j(u_t)) \\ &= u'_s \alpha'_s(u'_t) \end{aligned}$$

and

$$\begin{aligned}
\beta'_t(x') &= j(\beta_t(j(x'))) \\
&= j(u_t \alpha_t(j(x')) u_t^*) \\
&= j(u_t) \alpha'_t(x') j(u_t)^* \\
&= u'_t \alpha'_t(x') u'^*_t.
\end{aligned}$$

2) Writing  $U_t = u_t u'_t$ , notice first that if  $x \in M, x' \in M'$ , then

$$U_t \alpha_t(x) U_t^* = u_t \alpha_t(x) \bar{u}_t^* = \beta_t(x)$$

and

$$U_t \alpha'_t(x') U_t^* = u'_t \alpha'_t(x') u'^*_t = \beta'_t(x')$$

and in particular,

$$\beta_t(M) \vee \beta'_t(M') = U_t (\alpha_t(M) \vee \alpha'_t(M')) U_t^*$$

is a factor since  $\alpha$  is extendable; and hence, by Corollary 3.4, we deduce that  $\beta$  is extendable.

Finally, observe that

$$\begin{aligned}
U_{s+t} &= u_{s+t} u'_{s+t} \\
&= u_s \alpha_s(u_t) u'_s \alpha'_s(u'_t) \\
&= U_s \alpha_s^{(2)}(U_t)
\end{aligned}$$

and

$$\begin{aligned}
\beta_t^{(2)}(xx') &= \beta_t(x) \beta'_t(x') \\
&= u_t \alpha_t(x) u_t^* u'_t \alpha'_t(x') u'^*_t \\
&= U_t \alpha_t^{(2)}(xx') U_t^*.
\end{aligned}$$

□

**REMARK 5.12.** While the index of  $E_0$ -semigroups of type  $I_\infty$  factors has been well studied, we may now define the index of an extendable  $E_0$  semigroup  $\alpha$  of an arbitrary factor as the index of  $\alpha^{(2)}$ ; and we may infer from Proposition 5.11 that the index of an extendable  $E_0$ -semigroup of an arbitrary factor is an invariant of cocycle conjugacy.

**PROPOSITION 5.13.** *If  $\alpha = \{\alpha_t : t \geq 0\}$  (resp.,  $\beta = \{\beta_t : t \geq 0\}$ ) is an extendable  $E_0$  semigroup of a factor  $M$  (resp.,  $N$ ), then  $\alpha \otimes \beta = \{\alpha_t \otimes \beta_t : t \geq 0\}$  is an extendable  $E_0$  semigroup of the factor  $M \otimes N$ , and in fact,*

$$(\alpha \otimes \beta)^{(2)} = \alpha^{(2)} \otimes \beta^{(2)}.$$

*Proof.* The hypothesis is that  $\alpha_t(M) \vee J\alpha_t(M)J$  and  $\beta_t(N) \vee J_N\beta_t(N)J_N$  are factors, for each  $t \geq 0$ , while the conclusions follow from the definition of  $\alpha \otimes \beta$ .  $\square$

## 6 A non extendable $E_0$ -semigroup

Let  $\mathcal{H} = L^2(0, \infty) \otimes \mathcal{K}$ , where  $\mathcal{K}$  is a real Hilbert space and  $\mathcal{H}_{\mathbb{C}}$  be its complexification. The antisymmetric Fock space over  $\mathcal{H}_{\mathbb{C}}$  is defined as the direct sum of Hilbert spaces  $\{\wedge^n \mathcal{H}_{\mathbb{C}} : n = 0, 1, 2, \dots\}$ , where  $\wedge^0 \mathcal{H}_{\mathbb{C}} = \mathbb{C}$  and  $\wedge^n \mathcal{H}_{\mathbb{C}}$  is the antisymmetric subspace of the  $n$ -fold tensor product  $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ . We will denote this antysymmetric Fock space by

$$\mathcal{F}(\mathcal{H}_{\mathbb{C}}) = \mathbb{C}\Omega \oplus \mathcal{H}_{\mathbb{C}} \oplus (\mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}}) \oplus (\mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}}) \oplus \dots,$$

where  $\Omega$  is a fixed complex number with modulus 1.

Let  $\{e_1, e_2, \dots\}$  be an orthonormal basis for  $\mathcal{H}_{\mathbb{C}}$ ; then for fixed  $n$ ,  $\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n} : 1 \leq i_1 < i_2 < \dots < i_n\}$  forms an orthonormal basis for  $\wedge^n \mathcal{H}_{\mathbb{C}}$ .

For a fixed  $f \in \mathcal{H}_{\mathbb{C}}$  there is an unique operator  $a(f) \in \mathcal{B}(\mathcal{F}(\mathcal{H}_{\mathbb{C}}))$  which acts as follows:

$$\begin{aligned} a(f)\Omega &= f \\ a(f)(\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_n) &= f \wedge \xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_n, \quad \xi_i \in \mathcal{H}_{\mathbb{C}} \end{aligned}$$

The operator  $a(f)$  is called the left creation operator corresponding to the vector  $f \in \mathcal{H}_{\mathbb{C}}$ . These operators obey the *canonical anti-commutation relations* meaning:  $\mathcal{H}_{\mathbb{C}} \ni f \mapsto a(f) \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$  is a linear map satisfying

$$a(f)a(g) + a(g)a(f) = 0, \quad a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle 1$$

for all  $f, g \in \mathcal{H}_{\mathbb{C}}$ , where  $\langle , \rangle$  is an inner product on  $\mathcal{H}_{\mathbb{C}}$  and  $1$  is the identity operator in  $\mathcal{B}(\mathcal{H}_{\mathbb{C}})$ .

For any  $f \in \mathcal{H}_{\mathbb{C}}$  let  $u(f) = a(f) + a(f)^*$ , so the operator  $u(f)$  is a self-adjoint unitary for every unit vector  $f \in \mathcal{H}$ . Consider the von

Neumann algebra  $\mathcal{R} = \{u(f) : f \in \mathcal{H}\}''$  in  $\mathcal{B}(\mathcal{F}(\mathcal{H}_{\mathbb{C}}))$ , with vacuum state  $\rho$  (i.e.  $\rho(a) = \langle a\Omega, \Omega \rangle$  for  $a \in \mathcal{R}$ ). It is well-known that  $\mathcal{R}$  is the hyperfinite  $II_1$  factor with normal tracial state  $\rho$  and that its GNS space is  $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$  and  $\Omega$  is a cyclic and separating for  $\mathcal{R}$  and observe that  $\Omega = \widehat{1_{\mathcal{R}}}$ .

Let  $\{s_t\}_{t \geq 0}$  be the shift semigroup on  $\mathcal{H}_{\mathbb{C}}$ . Then for every  $t \geq 0$  there exist a unique normal, unital  $*$ -endomorphism  $\alpha_t : \mathcal{R} \mapsto \mathcal{R}$  satisfying

$$\alpha_t(u(f)) = u(s_t f) \forall f \in \mathcal{H}_{\mathbb{C}}.$$

(Although this is a well-known fact, we remark that this fact is in fact a consequence of Remark 2.2 (2).) Then  $\alpha = \{\alpha_t : t \geq 0\}$  form an  $E_0$ -semigroup on  $\mathcal{R}$ , which is called the **Clifford flow of rank  $\dim \mathcal{K}$** .

We want to prove that this Clifford flow is not extendable. By Corollary 5.10 it is sufficient to prove that if  $H_t = E^{\alpha_t} \cap E_t^{\alpha'}$ , then the set  $\{(t, \xi) : t > 0, \xi \in H_t \Omega\}$  is not a product system with respect to the family of unitary maps

$$v_{st} : H_s \Omega \otimes H_t \Omega \mapsto H_{s+t} \Omega,$$

given by  $v_{st}(T\Omega \otimes S\Omega) = TS\Omega$ , for  $S \in H_s, T \in H_t$ .

LEMMA 6.1. *With the above notation, if  $A \in H_t$ , then*

$$A\Omega = \sum \lambda_t^T(P_t^A)Q(P_t^A)\Omega$$

where  $P_t^A$  are appropriate subsets of  $\mathcal{P}_t$  of even cardinality, and  $\lambda(P_t^A) \in \mathbb{C}$ .

*Proof.* We have  $H_t = E_t^{\alpha'} \cap E^{\alpha_t} = [\mathcal{R}u_t] \cap \alpha_t(\mathcal{R})'u_t$  (by equations (5.9) and (5.10)). Now  $A \in H_t$  can be written as  $Tu_t$ , where  $T \in \alpha_t(\mathcal{R})'$ . Note that  $A\Omega = Tu_t\Omega = T\Omega$  since  $u_t\Omega = \Omega$ . Now choose orthonormal bases  $\{f_i\}_{i \in \mathcal{P}_t}$  of  $L^2(0, t) \otimes \mathcal{K}$  and  $\{g_j\}_{j \in \mathcal{F}_t}$  of  $L^2(t, \infty) \otimes \mathcal{K}$  (where  $\mathcal{P}_t$  and  $\mathcal{F}_t$  are totally ordered sets); the symbols  $P$  and  $F$  are meant to signify past and future respectively. Then clearly  $\{f_i\}_{i \in \mathcal{P}_t} \cup \{g_j\}_{j \in \mathcal{F}_t}$  is an orthonormal basis for  $L^2(0, \infty) \otimes \mathcal{K}$ .

For  $P_t = \{i_1 < i_2 < \dots < i_n\} \in \mathcal{P}_t$  and  $F_t = \{j_1 < j_2 < \dots < j_m\} \in \mathcal{F}_t$ , write  $Q(P_t) = u(f_{i_1})u(f_{i_2}) \cdots u(f_{i_n})$ ,  $Q(F_t) = u(g_{j_1})u(g_{j_2}) \cdots u(g_{j_m})$ , (with  $Q(\emptyset) = 1$ ). Then, observe (thanks to the CAR) that

$$Q(P_t)Q(F_t)\Omega = f_{i_1} \wedge \cdots \wedge f_{i_n} \wedge g_{j_1} \wedge \cdots \wedge g_{j_m},$$

and that consequently  $\{Q(P_t)Q(F_t)\Omega : P_t \subset \mathcal{P}_t, F_t \subset \mathcal{F}_t\}$  is an orthonormal basis for  $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ ; hence, any element  $\xi \in \mathcal{F}(\mathcal{H}_{\mathbb{C}})$  can be written as:

$$\xi = \sum \lambda(P_t, F_t) Q(P_t) Q(F_t) \Omega, \quad \lambda(P_t, F_t) \in \mathbb{C},$$

and in particular, we can write:

$$T\Omega = \sum \lambda(P_t, F_t) Q(P_t) Q(F_t) \Omega,$$

On the other hand as also  $Tu_t \in [Mu_t]$ , there exists a net  $\{m_\lambda\}_{\lambda \in \Lambda}$  in  $M$  ( $\Lambda$  some directed set) such that  $m_\lambda u_t$  converges strongly to  $Tu_t$ . So  $m_\lambda u_t m\Omega = m_\lambda \alpha_t(m)\Omega$  converges to  $Tu_t m\Omega = T\alpha_t(m)\Omega$  in norm, for  $m \in \mathcal{R}$ . In particular,  $m_\lambda Q(F_t^0)\Omega$  converges to  $TQ(F_t^0)\Omega$  in norm, since  $Q(F_t^0) \in \alpha_t(\mathcal{R}) \forall F_t^0 \subseteq \mathcal{F}_t$ .

Again,  $m_\lambda \Omega = m_\lambda u_t \Omega$  converges to  $Tu_t \Omega = T\Omega$  in the norm. So  $\rho_{Q(F_t^0)} m_\lambda \Omega$  converges to  $\rho_{Q(F_t^0)} T\Omega$ , where  $\rho_{Q(F_t^0)} = JQ(F_t^0)^*J$  so that  $\rho_{Q(F_t^0)}(m\Omega) = mQ(F_t^0)\Omega$ . Observe, then, that

$$\begin{aligned} Q(F_t^0)T\Omega &= TQ(F_t^0)\Omega \quad (\text{since } T \in \alpha_t(\mathcal{R})') \\ &= \lim m_\lambda Q(F_t^0)\Omega \\ &= \lim \rho_{Q(F_t^0)} m_\lambda \Omega \\ &= \rho_{Q(F_t^0)} T\Omega \\ &= \rho_{Q(F_t^0)} \sum \lambda(P_t, F_t) Q(P_t) Q(F_t) \Omega \\ &= \sum \lambda(P_t, F_t) Q(P_t) Q(F_t) Q(F_t^0) \Omega. \end{aligned}$$

So we conclude that

$$\begin{aligned} T\Omega &= \sum \lambda(P_t, F_t) Q(F_t^0)^* Q(P_t) Q(F_t) Q(F_t^0) \Omega \quad (\text{since } Q(F_t^0) \text{ is unitary}) \\ &= \sum \lambda(P_t, F_t) Q(F_t^0)^* Q(P_t) Q(F_t) Q(F_t^0) \Omega, \\ &= \sum (-1)^{\theta_{P_t, F_t}(F_t^0)} \lambda(P_t, F_t) Q(P_t) Q(F_t) \Omega, \end{aligned}$$

where  $\theta_{P_t, F_t}(F_t^0) = |P_t| |F_t^0| + |F_t| |F_t^0| - |F_t \cap F_t^0|$ . So finally we have,

$$\sum \lambda(P_t, F_t) Q(P_t) Q(F_t) \Omega = \sum (-1)^{\theta_{P_t, F_t}(F_t^0)} \lambda(P_t, F_t) Q(P_t) Q(F_t) \Omega,$$

whence  $(-1)^{\theta_{P_t, F_t}(F_t^0)}$  must be even for every finite ordered subset  $F_t^0$  of  $\mathcal{F}_t$ . It follows that  $P_t$  is even and  $F_t = \emptyset$ . So finally we have

$$T\Omega = \sum \lambda(P_t) Q(P_t) \Omega, \quad \lambda(P_t) \in \mathbb{C},$$

where  $P_t = \{i_1 < i_2 < \dots < i_{2n}\} \subseteq \mathcal{P}_t$ , thereby proving the lemma.  $\square$

**PROPOSITION 6.2.** *If  $\alpha$  is the Clifford flow, then for any fixed  $t > 0$ , we have*

$$H_t\Omega = [\{f_1 \wedge f_2 \wedge \dots \wedge f_{2n} : f_i \in L^2(0, t) \otimes \mathcal{K}, n \in \mathbb{N} \cup \{0\}\}] ,$$

where the empty wedge-product (i.e., the case when  $n = 0$ ) is interpreted as the vector  $\Omega$ .

*Proof.* The above Lemma implies the inclusion

$$H_t\Omega \subseteq [\{f_1 \wedge f_2 \wedge \dots \wedge f_{2n} : f_i \in L^2(0, t) \otimes \mathcal{K}, n \in \mathbb{N} \cup \{0\}\}].$$

To prove the reverse inclusion, it is enough to prove that

$$u(f_{i_1})u(f_{i_2}) \cdots u(f_{i_{2n}})e_t \in \alpha_t(\mathcal{R})' \cap J\alpha_t(\mathcal{R})'J, \quad (6.11)$$

since  $H_t = E_t^{\alpha t} \cap E_t^{\alpha'} = \mathcal{R}_1(t)'u_t \cap \mathcal{R}_1(t)u_t$  (by equations (5.9) and (5.10)). As  $\mathcal{R}$  is  $II_1$  factor so we have  $H_t = \alpha_t(\mathcal{R})'u_t \cap J\alpha_t(\mathcal{R})'Ju_t$ , where of course  $J$  is the modular conjugation for  $\mathcal{R}$ . It is clear that

$$u(f_{i_1})u(f_{i_2}) \cdots u(f_{i_{2n}})e_t \in \alpha_t(\mathcal{R})',$$

where, as before,  $\{f_i\}_{i \in \mathcal{P}_t}$  is an orthonormal basis for  $L^2(0, t) \otimes \mathcal{K}$  and  $\{g_j\}_{j \in \mathcal{F}_t}$  is an orthonormal basis for  $L^2(t, \infty) \otimes \mathcal{K}$ . Now observe, as a consequence of the CAR, that if  $\{h_i : i \in I\}$  is any orthonormal set in  $L^2(0, \infty) \otimes \mathcal{K}$ , then

$$\begin{aligned} u(h_n) \cdots u(h_1) &= (-1)^{n-1}u(h_1)u(h_n) \cdots u(h_2) \\ &= \dots \\ &= (-1)^{(n-1)+(n-2)+\dots+2+1}u(h_1)u(h_2) \cdots u(h_n) \\ &= (-1)^{\frac{n(n-1)}{2}}u(h_1)u(h_2) \cdots u(h_n), \end{aligned}$$

and hence

$$\begin{aligned} Ju(f_{i_1}) \cdots u(f_{i_{2n}})e_t J(g_{j_1} \wedge \dots \wedge g_{j_m}) &= Ju(f_{i_1}) \cdots u(f_{i_{2n}})Ju(g_{j_1}) \cdots u(g_{j_m})\Omega \\ &= Ju(f_{i_1}) \cdots u(f_{i_{2n}})u(g_{j_m}) \cdots u(g_{j_1})\Omega \\ &= u(g_{j_1}) \cdots u(g_{j_m})u(f_{i_{2n}}) \cdots u(f_{i_1})\Omega \\ &= u(f_{i_{2n}}) \cdots u(f_{i_1})u(g_{j_1}) \cdots u(g_{j_m})\Omega \\ &= (-1)^{n(2n-1)}u(f_{i_1}) \cdots u(f_{i_{2n}})e_t(g_{j_1} \wedge \dots \wedge g_{j_m}) \end{aligned}$$

Since the  $g_j$ 's were arbitrary and since the  $g_{j_1} \wedge \cdots \wedge g_{j_m}$  form an orthonormal basis for  $\text{ran } e_t$ , we find thus that  $J u(f_{i_1}) \cdots u(f_{i_{2n}}) e_t J = (-1)^{n(2n-1)} u(f_{i_1}) \cdots u(f_{i_{2n}}) e_t \in \alpha_t(\mathcal{R})'$  for each fixed  $n$ . That means  $u(f_{i_1}) \cdots u(f_{i_{2n}}) e_t \in J \alpha_t(\mathcal{R})' J$ . Therefore we have,

$$H_t \Omega = [\{f_1 \wedge f_2 \wedge \cdots \wedge f_{2n} : f_i \in L^2(0, t) \otimes \mathcal{K}, n \in \mathbb{N} \cup \{0\}\}].$$

□

Now we want to prove that  $\{(x, t) : x \in H_t \Omega\}$  is not a product system with the family of maps  $v_{st} : H_s \Omega \otimes H_t \Omega \mapsto H_{s+t} \Omega$  given by

$$v_{st}(A \Omega \otimes B \Omega) = AB \Omega,$$

where  $A \in H_s$  and  $B \in H_t$ . Let  $\{f_i\}_{i \in \mathcal{P}_t}$  be an orthonormal basis for  $L^2(0, t) \otimes \mathcal{K}$  and  $\{h_l\}_{l \in \mathcal{P}_s}$  is an orthonormal basis for  $L^2(0, s) \otimes \mathcal{K}$ . If  $A = Su_s \in H_s$ , then Lemma 6.1 allows us to write:

$$S \Omega = \sum \lambda(P_s^A) Q(P_s^A) \Omega, \quad \lambda(P_s^A) \in \mathbb{C},$$

and if  $B = Tu_t \in H_t$  we can write:

$$T \Omega = \sum \lambda(P_t^B) Q(P_t^B) \Omega, \quad \lambda(P_t^B) \in \mathbb{C}.$$

Now we have

$$\begin{aligned} v_{st}(Su_s \Omega \otimes Tu_t \Omega) &= Su_s T u_t \Omega \\ &= Su_s \sum \lambda(P_t^B) Q(P_t^B) \Omega \\ &= \sum \lambda(P_t^B) S(u_s Q(P_t^B)) \Omega \\ &= \sum \lambda(P_t^B) (u_s Q(P_t^B)) S \Omega \quad (\text{since } S \in \alpha_s(\mathcal{R})') \\ &= \sum \lambda(P_t^B) \lambda(P_s^A) (u_s Q(P_t^B)) Q(P_s^A) \Omega \\ &= \sum \lambda(P_s^A) \lambda(P_t^B) Q(P_s^A) (u_s Q(P_t^B)) \Omega, \end{aligned}$$

where the asserted commutativity in the last line stems from the CAR and the fact that  $P_s^A$  and  $P_t^B$  have even cardinality.

It is seen from equation (6.11) and the subsequent remarks, that if we set  $P_s^A = \{i_1 < i_2 < \cdots < i_{2m}\} \subseteq \mathcal{P}_s^A$  and  $P_t^B = \{j_1 < j_2 < \cdots < j_{2n}\} \subseteq \mathcal{P}_t^B$ , then  $Q(P_s^A) e_s u_s \in H_s$ , and similarly  $Q(P_t^B) e_t u_t \in H_t$ . Now, since

$$\begin{aligned} u_s Q(P_t^B) \Omega &= u_s (u(f_{j_1}) \cdots u(f_{j_{2n}})) \Omega \\ &= u(s_s f_{j_1}) \cdots u(s_s f_{j_{2n}}) \Omega \\ &= s_s f_{j_1} \wedge \cdots \wedge s_s f_{j_{2n}}, \end{aligned}$$

we find that

$$\begin{aligned}
& v_{st}(h_{i_1} \wedge \cdots \wedge h_{i_{2m}} \otimes f_{j_1} \wedge \cdots \wedge f_{j_{2n}}) \\
&= v_{st}(Q(P_s^A)\Omega \otimes Q(P_t^B)\Omega) \\
&= v_{st}(Q(P_s^A)e_s u_s \Omega \otimes Q(P_t^B)e_t u_t \Omega) \\
&= Q(P_s^A)u_s Q(P_t^B)\Omega \\
&= h_{i_1} \wedge \cdots \wedge h_{i_{2m}} \wedge s_s f_{j_1} \wedge \cdots \wedge s_s f_{j_{2n}}.
\end{aligned}$$

Clearly  $v_{st}$  is not onto, since range of  $v_{st}$  is orthogonal to vectors of the form  $f \wedge g$ , where  $f \in L^2(0, s) \otimes \mathcal{K}$  and  $g \in L^2(s, s+t) \otimes \mathcal{K}$ . So  $\{(x, t) : x \in H_t\Omega\}$  can not be a product system with the family of maps  $\{v_{st} : s, t > 0\}$ , since they are not unitary.

The Clifford flows of the hyperfinite  $\text{II}_1$  factor are closely related to another family of  $E_0$ -semigroups, called the CAR flows. (We should remember these are CAR flows on type  $\text{II}_1$  factors, not to be confused with the usual CAR flows on the algebra of all bounded operators on the antisymmetric Fock space.) We recall the definition of CAR algebra and some facts regarding the GNS representations of CAR algebras given by quasi-free states.

For a complex Hilbert space  $K$ , the associated CAR algebra  $CAR(K)$  is the universal  $C^*$ -algebra generated by a unit 1 and elements  $\{b(f) : f \in K\}$ , subject to the following relations

- (i)  $b(\lambda f) = \lambda b(f)$ ,
- (ii)  $b(f)b(g) + b(g)b(f) = 0$ ,
- (iii)  $b(f)b^*(g) + b^*(g)b(f) = \langle f, g \rangle 1$ ,

for all  $\lambda \in \mathbb{C}$ ,  $f, g \in K$ , where  $b^*(f) = b(f)^*$ .

Given any contraction  $A$  on  $K$ , there exists a unique quasi-free state  $\omega_A$  on  $CAR(K)$  satisfying

$$\omega_A(b(x_n) \cdots b(x_1)b(y_1)^* \cdots b(y_m)^*) = \delta_{n,m} \det(\langle Ax_i, y_j \rangle),$$

where  $\det(\cdot)$  denotes the determinant of a matrix. Let  $(H_A, \pi_A, \Omega_A)$  be the corresponding GNS triple. Then  $M_A = \pi_A(CAR(K))''$  is a factor.

Here onwards we fix the contraction with  $A = \frac{1}{2}$ , then  $M_A = \mathcal{R}$  is the hyperfinite type  $\text{II}_1$  factor. We define the CAR flow on  $\mathcal{R}$  as follows.

As before let  $K = L^2((0, \infty), k)$  with dimension of  $k$  being  $n$ , and  $S_t$  be the shift on  $K$ . There exists a unique  $E_0$ -semigroup  $\{\alpha_t\}$  on  $\mathcal{R}$  satisfying

$$\alpha_t(\pi(b(f))) = \pi(b(S_t f)) \quad \forall f \in K.$$

This  $\alpha$  is called the CAR flow of index  $n$  on  $\mathcal{R}$ .

We recall the following proposition from [Alev] (see proposition 2.6).

**PROPOSITION 6.3.** *The CAR flow of rank  $n$  on  $\mathcal{R}$  is conjugate to the Clifford flow of rank  $2n$ .*

We point out an error in [ABS] in the following remark.

**REMARK 6.4.** *In section 5, [ABS], it is claimed that CAR flows of any given rank is extendable. In fact a ‘proof’ is given, for any  $\lambda \in (0, \frac{1}{2}]$  with  $A = \lambda$ , that the corresponding  $E_0$ -semigroup on  $M_A$  is extendable. (When  $\lambda \neq \frac{1}{2}$  they are type III factors.) But we have proved that Clifford flows are not extendable. This consequently implies, thanks to proposition 6.3, and the invariance of extendability under cocycle conjugacy, that CAR flows on the hyperfinite type  $II_1$  factor  $\mathcal{R}$  are not extendable.*

*As far as we see, their assertion, in theorem 4 of section 5, that  $\theta'(\Gamma \otimes \Gamma) = \Gamma \otimes \Gamma$  is not correct, where  $\Gamma$  is defined by  $\Gamma\Omega = \Omega$  and  $\Gamma\pi(b(f)) = -\pi(b(f))\Gamma$  for all  $f \in K$ . In fact, in the case of CAR flows  $\alpha_t(\Gamma)$  will satisfy  $\alpha_t(\Gamma)\pi(b(S_tf)) = -\pi(b(f))\alpha_t(\Gamma)$  for all  $f \in K$  and  $\alpha_t(\Gamma)\pi(b(f_t)) = \pi(b(f_t))\alpha_t(\Gamma)$  for all  $f \in (S_tK)^\perp$ .*

*In fact we suspect that CAR flows for all  $\lambda \neq \frac{1}{2}$  are also not extendable.*

Let  $\Gamma_f(K)$  be the full Fock space associated with a Hilbert space  $K$ . Define the operator  $s(f) = \frac{l(f)+l(f)^*}{2}$  on  $\Gamma_f(L^2((0, \infty); k^{\mathbb{C}}))$  where

$$l(f)\xi = \begin{cases} f & \text{if } \xi = \Omega, \\ f \otimes \xi & \text{if } \langle \xi, \Omega \rangle = 0. \end{cases}.$$

The von Neumann algebra  $\Phi(k) = \{s(f) : f \in L^2((0, \infty); k)\}''$ , is isomorphic to the free group factor  $L(F_\infty)$  and the vacuum is cyclic and separating with  $\langle \Omega, x\Omega \rangle = \tau(x)$  (see [DV]). There exists a unique  $E_0$ -semigroup  $\gamma$  on  $\Phi(k)$  satisfying

$$\gamma_t(s(f)) := s(S_tf) \quad (f \in k, t \geq 0).$$

This is called the free flow of rank  $\dim k$ .

**REMARK 6.5.** *Let  $\gamma$  be free flow of any rank. It is proved in [OV] that*

$$E_t^\gamma \cap E_t^{\gamma'} = \mathbb{C}.$$

*This means, if the free flow  $\gamma$  is extendable, then the product system associated with the extended  $E_0$ -semigroup is one dimensional. This*

implies that the extended  $E_0$ -semigroup is cocycle conjugate to part of an automorphism group. But this is not possible since  $S_t \in E_t^\gamma \cap E_t^{\gamma'}$  is an isometry.

Also free flows provide examples to show that the converse of Lemma 5.9 is not true. For free flows we do have the family  $(E_t^\gamma \cap E_t^{\gamma'} : t \geq 0)$  forms product system, but still they are not extendable.

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